**Cumulative Area Test (CAT)**

A randomization test for measuring ranking ability of prospect appraisal predictions.

The CAT statistic is given by:

$$CAT = \sum_{i=1}^{n} \sum_{k=1}^{i} x_k - \tfrac{1}{2} n(n+1)\bar{x}$$

The sum of CAT values under all n! permutations is:

$$\begin{array}{ll}
x_1 & -\ \bar{x} \\
x_1 + x_2 & -\ 2\bar{x} \\
x_1 + x_2 + x_3 & -\ 3\bar{x} \\
x_1 + x_2 + x_3 \cdots + x_n & -\ n\bar{x} \\
\text{- - - - - - - - - - - - - - - - - - - -} & + \\
nx_1 + (n-1)x_2 + \cdots + x_n - \tfrac{1}{2} n(n+1)\bar{x} &
\end{array}$$

**where all indexes are permuted.**

The above expansion of the CAT statistic under permutation forms a table with n! lines. The first term in each line can have the index 1 in only (n-1)! cases, it occurs in the second term also in (n-1)! cases, etc. The sum therefore becomes:

$$\sum_{m=1}^{n!} CAT = (n-1)!\,n\sum_{i=1}^{n} x_i + (n-1)!(n-1)\sum_{i=1}^{n} x_i + \cdots - \tfrac{1}{2} n(n+1)\bar{x}n!$$

Division by n! results in:

$$\mu = \frac{(n-1)!\,\tfrac{1}{2} n(n+1)\sum_{i=1}^{n} x_i - \tfrac{1}{2} n(n+1)\bar{x}n!}{n!} =$$

$$\mu = \tfrac{1}{2} n(n+1)\bar{x} - \tfrac{1}{2} n(n+1)\bar{x} = 0$$

as expected from symmetry considerations.

The variance of CAT under the n! permutations is derived by calculating the second moment. We count all permutations of:

$$\left[nx_1 + (n-1)x_2 + \cdots x_n\right]^2$$

Expanding the expression and writing out all the possible permutations of the indexes gives a table with n! rows. In each row there are n squares (e.g. $x_1^2$) and ½$n$($n$-1) cross-products (e.g. $x_1 x_2$). The sum of all the squares under permutation becomes:

$$(n-1)!\,n^2\sum_{i=1}^{n} x_i^2 + (n-1)!(n-1)^2\sum_{i=1}^{n} x_i^2 + \cdots + (n-1)!\sum_{i=1}^{n} x_i^2 =$$

$$(n-1)!\,\tfrac{1}{6}n(n+1)(2n+1)\sum_{i=1}^{n} x_i^2$$

The sum of the cross products is derived by considering a cross-product term in a particular position in a line. A particular combination of indexes (e.g. 1 and 2) in a particular term occurs in only (n-2)! lines out of the n! lines in total. For n=4 such a line in the table would look as follows:

$$24x_1x_2 + 16x_1x_3 + 12x_1x_4 + 8x_2x_3 + 6x_2x_4 + 4x_3x_4$$

The line contains cross-products proper and coefficients. The coefficients are forming all possible products of $2\cdot i\cdot j$ **where** $i \neq j$. So for a particular cross-product $x_i x_j$ the sum of the coefficients can be expressed as:

$$(n-2)!\,2\left[\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i_i^2\right]$$

and the sum of the cross-products as:

$$(n-2)!\,2\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right]$$

making the sum :

$$2(n-2)!\,\frac{3n^2(n+1)^2 - 2n(n+1)(2n+1)}{24}\left[\left(\sum x\right)^2 - \sum x^2\right]$$

Taking the sum over n! of the squares and the cross products and dividing by n! gives:

$$\mathrm{var}(CAT) = \frac{n(n+1)}{12}\sum x^2 + \frac{(n+1)(3n+2)}{12}\left(\sum x\right)^2 - \tfrac{1}{4}n^2(n+1)^2\bar{x}^2$$

$$\mathrm{var}(CAT) = \tfrac{1}{12}n^2(n+1)\,\mathrm{var}(x)$$